Student Academic Success in Linear Algebra in an Open Distance Learning Environment[10]

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ABSTRACT

Academic success in first year university mathematics in been problematic for decades and mathematics educators keep looking for the causes. What has been universally agreed, is that the theoretical and abstract nature of mathematics plays a role. A module on Linear Algebra at an Open Distance eLearning (ODeL) institution, was identified for investigation due to very high dropout and failure rates. This article concentrates on identifying the types of knowledge (e.g., procedural, conceptual, strategic, schematic and declarative) necessary for academic success in the subject. Using the literature, a conceptual framework is developed to classify students' answers into the various types of knowledge. The research question asks what types of knowledge contributes to academic success in Linear Algebra. Script analysis is used to answer the research question. The results showed that lack of the necessary declarative knowledge which forms the basis for the other forms of knowledge as well as procedural knowledge were the main causes of the resulting misconceptions and errors. It was established that students were more engaged in surface learning rather than deep learning that results in conceptual understanding and acquisition of conceptual knowledge.

Keywords: Linear Algebra, ODeL, types of knowledge, mathematics content knowledge, procedural knowledge, conceptual knowledge

INTRODUCTION

Linear Algebra has been recognised as an important field of mathematics due to its applications in other fields of mathematics such as differential equations, analysis and probability among others. It is also a basis for understanding many topics in physics, chemistry, biology engineering as well as economy (Çelik, 2015). Investigating academic success in Linear Algebra thus, cannot be dissociated from that of academic success in mathematics. Several researchers (Britton & Henderson, 2009; Doneveska-Todorova, 2014; Rensaa & Vos, 2018; Soylu & Işik, 2008) have also highlighted the inclination of mathematical acquisition to dependence on Linear Algebra. The fact that the nature of mathematics is one of abstractness and theoretical (Britton & Henderson, 2009) implies that the teaching and learning (T & L) of the subject employs a particular approach; its own psychology of learning. This was recognised

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by Richard Skemp (1987) back in the 1970s who introduced 'The psychology of learning Mathematics'. Other researchers also related the psychology of learning mathematics to linear algebra (Dorier & Sierpinska, 2002; Hillel, 2000; Sierpinska, 2000; Doneveska-Todorova, 2014). The researchers delineated that teaching that promotes deep learning as compared to surface learning encourages conceptual understanding. This approach to teaching mathematics thrives on students' acquisition of conceptual knowledge (knowledge of concrete and abstract concepts and their relationships) which is a synthesis of other forms of knowledge (e.g., procedural, declarative, schematic, metacognitive and strategic). Furthermore, very little research has been conducted on the effect of these different forms of knowledge in the learning of Linear Algebra in particular. This article asserts that we speak of knowledge of Mathematics we refer to conceptual knowledge with understanding. Conceptual understanding (relational understanding-Skemp, 1987) leads to conceptual knowledge which is the knowledge of the definitions and applications of the concept in varies situations and circumstance (Luneta 2015). Concept learning combined with principles that govern them leads to the structure of knowledge (Giannakopoulos, 2017; Skemp, 1987). Depending on how, when, why and where such knowledge is used, different names are assigned to it, such as procedural, declarative, conceptual, strategic and schematic. Combining knowledge with cognitive skills enables students to solve problems.
The study is part of a longitudinal research project conducted at an Open Distance Learning (ODeL) institution, to improve academic success. An Action Research approach was used. It was exploratory by nature as it aimed at establishing the types of knowledge necessary for the teaching and learning of Linear Algebra. Linear Algebra was chosen due to the high failure rates and the desire of the researcher to improve his instructional approach that results in optimised learning outcomes.

The specific research question that we attempted to answer is:

> What types of knowledge are required for first year students to excel in a Linear Algebra at an ODeL institution?

To answer this question, the following sub-questions were also important to address: it is necessary to answer the following sub-questions:

- What is the connection between the different types of Mathematics knowledge?
- Of the knowledges necessary to acquire mathematical understanding which was the most prominent among first year students studying linear algebra?

We envisaged that by identifying the types of knowledge students possess (or lack of it), the approach and the instruments used would serve as a diagnostic tool that can be used to improve learning outcomes. It is also important to note that these types of knowledge (or dimensions of knowledge as Krathwohl, 2002 calls it), declarative, conceptual, procedural, strategic, and schematic) are directly related to Bloom's (1979) taxonomy.

LITERATURE REVIEW

In the teaching and learning of Mathematics there is the teacher, the student, the environment and the content to be learned. As this study concentrates on the content to be learned, acquisition of knowledge depends on a number of factors. Research on Linear Algebra identified gender? sex, teaching style, and prior knowledge (Robert, 1017; Soylu & Işik, 2008), teaching and learning approaches, problem solving and self-efficacy (Ferryansyah, Widyati., & Rahayu, 2018; Murray, 2013; Orhun, 2012), cognition, attitude towards linear algebra, and student attributes (Ferryansyaand, et al., 2018). It has also been accepted by many researchers that the T & L learning of linear algebra is difficult for the teacher and the student alike (Britton & Henderson, 2009; Çelik, 2015; Dorier & Sierspinska, 2002; Ferryansyah et al., 2018R; Ferryansyah et al., 2018; Robert, 2017; Wood et al, 2002). Dorier and Sierspinska (2002) and Britton and Henderson, (2009) further assert that the difficulty lies on the nature of mathematics in general due to its abstractness and theoretical nature. Linear algebra tends to be difficult as it demands high levels of thinking due to its formal nature. Dorier (2002) called it the *obstacle of formalism* and teachers and students alike recognise formalism to be an obstacle to learning. Formalism demands the learning of concepts for conceptual understanding rather than computational algorithms driven by procedural knowledge (Britton & Henderson, 2009). The student is faced with a plethora of definitions, new symbols and theorems and has to operate wholly at an abstract level.

The formal nature of linear algebra is characterised by modes of description (language, such as geometric, arithmetic, algebraic) (Hillel, 2000) and these require certain modes of thinking that is characterised by cognitive flexibility, trans-object level of thinking, theoretical as opposed to practical thinking, analytic-arithmetic and analytic-structural modes of thinking (Dorier & Sierpinska, 2002: Sierpinska, 2000). Each mode of description can be associated with one or more modes of thinking (Donevska-Todorova, 2014). For a detailed discussion on these two modes the reader can refer (Dorier, 2002; Dorier & Sierpinska, 2002; Hille, 2000; Sierpinska, 2000). These modes of description and thinking assume that different types of knowledge exist in the cognitive structure. The successful learning of any new concept depends on the ability of the learner to connect the concept to the existing cognitive structure and presupposes that such a structure was formed from deep learning (relational) (Skemp, 1987), learning with understanding as opposed to surface or instrumental learning (Rensaa & Vos, 2018; Skemp, 1987). For Skemp (1987: 29) understanding means "to assimilate [something] into an appropriate [existing] schema". Accepting that learning of Linear Algebra's nature demands learning of concepts, concept formation and acquisition combined with principles give rise to structures of knowledge (Skemp, 1987). This knowledge mingled with cognitive skills on the make up the cognitive structure also known as conceptual structure. It is the conceptual structure that has evolved from conceptual understanding and appropriate application of concept definitions that are procedurally used to solve mathematical problems and linear algebra (Luneta 2015). In order to effectively teach and learn mathematics Krathwohl (2002) developed a two-dimensional model of knowledge dissemination and acquisition. One knowledge dimension deals with Bloom's revised taxonomy aligned to teaching and the other of knowledge dimension aligned to learning (Rensaa & Vos 2018; Shavelson, Ruiz-Primo & Wiley 2005).

In mathematics knowledge acquisition has been identified to occur at different levels and in different forms (Luneta 2015). Researchers have classified mathematical knowledge into forms or types namely - declarative, conceptual, procedural, strategic, schematic, conditional, metacognitive and situational (Rensaa & Vos, 2018; Rittle-Johnson & Schneider, 2015; Krathwohl, 2002; Shavelson et.al., 2005; Soylu & Işik, 2008)

We attempt to shed some light into the differences in the meanings given to the various types of knowledge. We start from a psychological perspective of mathematical knowledge acquisition as espoused by Skemp's (1987) who asserts that mathematical knowledge acquisition and development of concepts is from the lowest level to the highest abstraction level. Skemp (1987) states that a concept is acquired firstly at a low level, like the name of a concept (e.g., number 3, three) (surface learning). Through further abstractions the learner understands the concept and can apply it successfully to objects or item to mean or represent the number three. This knowledge of the concepts forms the *knowing that* level (Skemp, 1987: p. 115) which a number of researchers (Krathwohl, 2002; Rittle-Johnson & Schneider, 2015; Shavelson et.al., 2005; Soylu & Işik, 2008) called declarative knowledge (knowledge of facts, definitions, descriptions). This knowledge could be a word/ concept or a 'packet of concepts', called schema. The learning of concepts and their relationships to a point of defining and applying them to different situations gives rise to conceptual knowledge, which is defined as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information (Hiebert & Lefevre, 1986; Rittle-Johnson & Schneider, 2015; Wang, 2015).

Conceptual knowledge is seen as an integrated and functional grasp of mathematical ideas (Kilpatrick, Swafford & Findel, 2001 cited by Rensaa & Vos, 2018) and does not have a linear sequential character like the procedural knowledge (Donevska-Todorova, 2014). From the declarative knowledge we derive an appropriate plan of action which we may call *knowing how* (Skemp, 1987: p. 115) and a number of authors (Donevska-Todorova, 2014; Rittle-Johnson & Schneider, 2015) called that procedural knowledge. Hiebert and Lefevre (1986: p. 3) define procedural knowledge as consisting of mathematical language and "rules, algorithms or procedures used to solve mathematical tasks" or knowledge of procedures, a series of steps, or actions to accomplish a goal (Rittle-Johnson & Schneider, 2015). At this point a controversy arises that Rittle-Johnson and Schneider, (2015: p. 1126) called it the '*concepts-first*' or '*procedures-first* approach. They concluded that the relationship between procedural and conceptual knowledge is bidirectional and called it the '*iterative view*'. That is, as conceptual knowledge increases it leads to increase in procedural knowledge and vice versa since the initial knowledge could be conceptual or procedural (Rittle-Johnson & Schneider, 2015). Other researchers (Donevska-Todorova, 2014; Rensaa & Vos, 2018) have aligned the two knowledges to Linear Algebra. For instance, Donevska-Todorova (2014) provide an example of procedures-first and explain that by carrying out the Gaussian algorithm by using one's a procedural understanding it can be applied to solving a system of linear equations or finding an inverse of a matrix, thus linking it to other concepts which in this context can be regarded as conceptual understanding. In another example procedural knowledge is necessary to calculate the dot product of two vectors according to formula and conceptual understanding is involved when the dot product is used to calculate projections of vectors and the trigonometric function cosine, so to interpret the obtained scalar geometrically.

Rensaa and Vos (2018) stress that procedural and conceptual knowledge are related to students' learning and thinking and not to teaching and that conceptual knowledge confers benefits above and beyond having procedural skills. They further argue that linking procedural and conceptual knowledge provide computational shortcuts, ensure fewer errors and reduce forgetting. However, they warn us about confusing procedural knowledge with procedural fluency which is the result of conceptual understanding. On the other hand, conceptual knowledge could be superficial if it is constructed using weak schemas which are mainly related to primary level concepts. Renasaa and Vos (2018) further agree that procedural knowledge could be exhibited by the students if they have good mathematical skills in using formulae, algorithms and symbols, void of any understanding. However, interpreting and applying concepts, translating between verbal and visual and formal relationships such as the use of different representations in linear algebra is an indication of conceptual understanding. There are also misconceptions and errors that are associated with the acquisition of mathematical knowledge. Misconceptions result from lack of conceptual understanding, and they occur in the learner's mind and displayed as errors in spoken or written form by the learners (Luneta 2015). Luneta (2015) points out that misconceptions can result in several forms of errors (conceptual, procedural, practical, slipups/common mistake) depending on the severity of the misconceptions. Conceptual errors are those related to the lack of understanding of the concepts, resulting errors that show lack knowledge of the concept whereas procedural errors are errors related to use, adoption or display of wrong procedures and formula (Luneta 2015). Errors are therefore as a result of misconceptions.

The above exposition on declarative, conceptual and procedural mathematical knowledge highlights a very important aspect in acquisition of mathematical content knowledge. Declarative knowledge forms the basis of the other two types and the bidirectionality of conceptual and procedural is accepted by researchers such as Shavelson et al. (2005). But for declarative knowledge, schematic and strategic knowledge feature explicitly in research on linear algebra (Shavelson et al., 2005). The former deals with why we use certain concept (conceptual knowledge) or procedure (procedural knowledge). The latter deals with when, where and how the knowledge is used. For example, choosing the best procedure could save time in an exam, like a student who knows only to expand a determinant according to rows to apply it in a 5x5 determinant. Adding these two types of knowledge and using Shavelson et al. (2005)

conceptual framework we develop a new conceptual framework (see Figure 1) by connecting the types of knowledge to mathematical content and using the knowledge and various skills to solve mathematical problem. It is important to mention that,

Figure 1:
Mathematics content knowledge (MCK)

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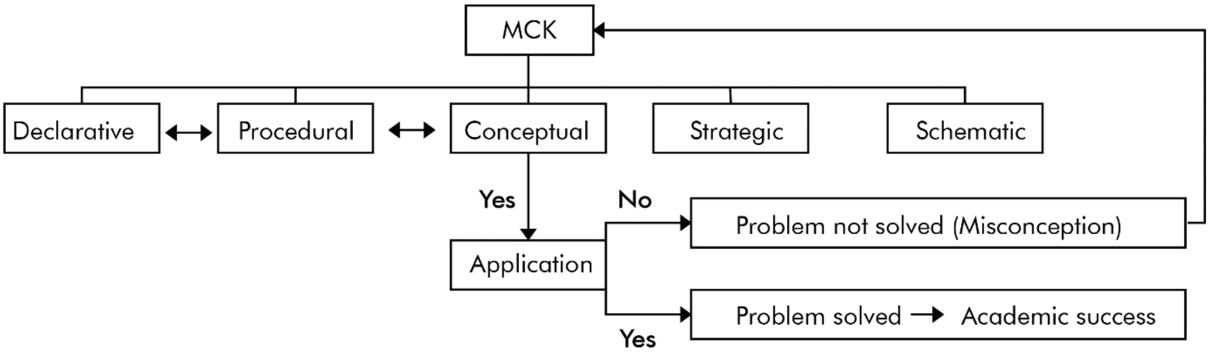

Shavelson et al. (2005) other than the dimensions of knowledge there is also the second dimension that of the 'extent', how proficient one is in any type or the level that the different types are acquired. In Figure 1, mathematics content knowledge comprises of different types of knowledge as identified by the literature. This knowledge is applied in solving problems. Successful application leads to academic success, while unsuccessful application (Luneta, 2015) implies misconceptions were formed during the use of one type of knowledge or the other. This framework forms the assessment instrument to analyse the existence or non-existence of one or more types of knowledge. It must be stressed here that the last two types of knowledge are not necessary to solve a problem but contribute to the quality of the solution. If we aim for deep learning, learning of concepts (concrete or abstract) should be goal-directed learning, learning with understanding (Skemp, 1987).

RESEARCH DESIGN

We attempt to develop a blueprint to answer the research question:

> What types of knowledge are required for first year students to excel in a Linear Algebra at an ODeL institution?

The literature review on the main constructs, reinforced the idea that the research problem is not only complex but that the interrelatedness between the constructs exacerbates that complexity. The research design then needs careful consideration so that the problem can be unpacked in order to solve it. Babbie and Mouton (2011) state that research designs usually fall into one of the three categories: Experimental, quasi-experimental, and non-experimental. These subsequently are divided into other sub-categories. In this particular case it is a non-experimental design. As it involves processing of marks from final examination scripts, the Quantitative method was followed. The data used was from students who sat for the final examination in linear algebra. In an ODeL education, where there are no face-to-face classes, all communication takes place either through electronic media (Learning Management System (LMS) (telephone, emails and pod casts). As a result, one way to assess students' performance is through their written responses to questions. The exam scripts were the primary data set and were used to capture students' mathematical knowledge, way of thinking, and problem-solving skills. Wood, Smith, Petocz and Reid (2002) state using examination results and scripts is practical and cost effective, objective and guaranteed quality assurance.

Data collection and analysis

The examination paper was divided into four questions, each question (Qi) covering a certain part of the syllabus. The concepts examined were: matrices/ determinants (Q1), linear systems (Q2), vectors (Q3) and complex numbers (Q4). Two sets of data were obtained. The first set, sample A, a purposive sample, comprised of the whole class of 26 students that sat for the final exam. This set of data assists us in forming an overview of the success rate of this class based on the numerical data obtained from the marks awarded to each student in the various questions of the examination paper and the concepts involved. Of the 35 students that were registered for linear algebra 5 dropped out and 4 did not qualify for the exam. Of the 26 that sat for the final examination 8 (31%) failed and only 1 re-registered. That meant the dropout rate in linear algebra was 37% (13 students) as 13 of the 35 students did not register in 2019.

The second set of data, sample B, also a purposive sample, aimed at identifying the types of knowledge demanded by each question. It comprised of 15 students and 5 scripts were randomly selected from 3 groups: Group A with an exam mark of 75% or more, Group B with a mark between 50% and 74% and Group C with a mark less than 50%. Students were assigned numbers 1-15 for ethical reasons (according to rank, thus student (5) refers to student number 5, ranked 5th). This assists us to refer to the same student in different situations. The various questions and sub-questions were analysed and the different types of knowledge necessary to solve the problems were identified and tabulated.

We would like to explain here that different questions that assessed diverse types of knowledge were used to assess their existence on nonexistent in the students' answers. The five criteria used below were used and can be used in any part of mathematics. Combining research done on types of knowledge and research done on linear algebra, these criteria are all connected to modes of description and modes of thinking discussed in the literature review. Thus, various sections of the linear algebra paper used here are connected to either or both modes and the learning of mathematics in general.

1. Declarative knowledge (D) (facts/ definitions/ formula)
2. Conceptual knowledge (C) (correct application, transformations, proofs)
3. Procedural knowledge (P) (performing procedures correctly)
4. Strategic knowledge (St) (choosing the most appropriate method)
5. Schematic knowledge (Sc) (evidence that the student knows why a particular method was chosen).

After analysis of scripts, it became apparent that St and Sc were not prevalent in any question. Using the memo of the examination paper each necessary step is classified accordingly. It is possible that a step could contain one or more types of knowledge. Therefore, a new memo is developed using the types of knowledge. The validity and reliability of such a memo will depend on the correlation between the marks allocated by the two memos. In certain cases, adjustments can be made in the new memo to ensure the highest correlation, since the new instrument used is not an official way of marking, but it will be used as a part of a diagnostic tool to improve teaching practice.

The two sets of data that were collected contain sample A with the students' numbers and the total marks of each sub-question were recorded. Thus, we have sub-questions 1a, 1b, 1c, and 1d and so on. Then for sample B, each step of the memo was classified in terms of the types of knowledge involved to solve the problem. Thus, if in step 1 we need the formula $u.v = a$ then a (D) is assigned. For calculation a (C) is assigned and so on. Descriptive statistics is used for process both samples followed by analysis of the students' answers.

The two sets of data are represented in Table 1. Table 1 shows the performance of students who sat for the exam with the top 15 achievers (Sample B (Avg, B, N=15)), and the percent of the group average, the median and the standard deviations were calculated. Then NS represents the number of students attempting a particular question followed by performance averages (Avg, P, D, C). Finally, the averages of sample A (Avg, B, N=26).

Table 1:
Students' performance in the examination paper (Samples A and B)

	Matrices/ Det/nants					Linear transf							Vectors					Compl.Nos					
Questions	1a	1b	1c	1d	Tot	2a	2b	2c	2d	2e	Tot	3a	3b	3c	3d	3e	Tot	4a	4b	4c	4d	4e	Tot
Avg.B,N-=15	3,7	4,1	6,2	8,1	22	4,7	4,4	2	1,6	2,7	15	4	4,3	2,3	4,5	4,1	18	2,7	5,2	4	1,4	4,4	11
%	98	87	94	96	94	100	67	40	24	89	62	85	77	83	84	64	78	79	87	48	57	30	48
Median	4	5	7	9	23	5	5,5	1	20	3	14	4,5	4,8	3	5	5	20	3	6	5	2	3	10
StdDev	0,8	1,6	1,8	1,8	3,8	0,6	2,9	2,2	45	0,7	5,5	1,5	1,9	1,2	1,3	2,2	6,2	0,7	1,9	1,8	0,9	4,2	9,1
NS	100	100	100	100		100	100	94	73	90		100	100	100	94	87		80	66	66	87	62	
%P	98	89	92	96	94	100	70	52	20	86	66	86	85	90	89	65	83	85	90	38		30	61
%D	97	84	95	90	92	100	71	48	31	84	67	84	79	91	90	60	81	90	86	60	57	32	65
%C	95	84	90	95	91	100	55	30	20	82	57	85	68	70	78	55	71	70	78	44		30	56
Avg (P,C.D)	97	86	92	94		100	65	43	24	84		85	77	84	86	60	78	82	85	47	57	31	60
Avg.A,N-=26	88	74	83	85	82	75	46	32	30	85	46	59	58	69	80	57	56	80	47	69	79	40	38

Performance in various questions by the class (Sample A)

For the whole class (sample A), students performed worst in Complex numbers (38%), followed by Linear Transformations (46%) then Vectors (56%) and last in Matrices/ Determinants performed best (82%). There were some sections where the students performed the worst, with averages ranging from 30% to 58%. For example, in the linear transformations section questions 2c and 2d dealt with proofs. No student that failed attempted these two questions). In question 2c (class average 32%) the students had to prove if A is a matrix and if $A^2 = A$ prove that det(A) = 0 or det(A)=1. For question 2d (class average 30%) students had to prove that two matrices are similar iff $A = PBP^{-1}$ where P is an invertible matrix and then prove that if A and B are similar then det(A) = det (B). In question 2b (class average 46%), the students were asked to find the determinant of a 4x4 matrix. Thus, 2c and 2d require both conceptual and procedural knowledge.

In the Vectors section students were asked to determine the dot product of 3-D vectors and the angle between them (question 3a, class average 59%), the orthogonal projection (question 3b, class average 58%) and the vector component if that vector is orthogonal to another. All students who attempted this question were successful as long as they knew the definition (formula) for an orthogonal projection (which is declarative knowledge). For question 3e where 3 points were given and asked to determine the equation of the plane passing through them, 87% attempted the question and averaged 67%. No student got 100% for this question. In the Complex numbers section, the class performed worst in the last question (4e, class average 40%) where a complex number is in standard form is given, and all the square roots must be calculated, and the answer must be in polar form. Here again P and C are involved. In question 4b (class average 46%) the students were given a Complex number in standard form, a +bi, to express it in polar form and another in quotient form to express it in the standard form. For question 1 (class average 82%) students performed worst in reducing a 4x4 matrix to row echelon form. Reduction to echelon form requires poor procedural knowledge.

Performance in various questions by the top 15 students (Sample B)

This group (sample B) was chosen to investigate the existence/ nonexistence of the types of knowledge by doing script analysis. This group's patterns of performance were mostly that of the class. For example, they equally performed worse in questions 2c and 2d (43% and 24%, respectively) and 4e (31%). In the other questions discussed above for the whole class this group performed relatively better. Having marked the papers according to D, C, and P, a 98% correlation was obtained between this method of marking and that of the official marker. It must be reiterated that this method of assessment is for diagnostic purposes and not for marking exam papers. Another interesting observation in Table 1 is the percentage of the students (sample B) that attempted some questions. The lowest occurred in Complex numbers and question 2d where they had to prove certain statements. The top 8 students attempted 100% of the questions. The standard deviations were relatively low (0,6 the lowest) and only in the question 4e it was more than 2,5.

Furthermore, the following criteria were used to analyse the scripts further.

1) For question 1 where the students performed very well question 2b was chosen which carried the lowest average mark and contains matrix/ determinant operations which are used in other parts of question 1 as well as in question 2, where similar errors appeared.
2) Questions 2c ,2d 3b, 3e, 4c, and 4e were chosen because the students performed the worst like the whole class.

It was decided to group the questions into various categories which shared common concepts. Examining the question paper, it was found that a number of questions (26%) (Group A) of the paper comprises of the use of a formula, substitution into the formula and performing operations. For example, use Cramer's rule, dot-product, angle between vectors, cross-product, area of a parallelogram, orthogonal projection, and the distance between two points. Performing operations on various concepts, e.g., matrix/ determinant operations, vectors and complex numbers was 32% (Group B). Involving formulae (D), performing operations (C) and following the correct path (P) is an indication that lack of any of the three, the problem cannot be fully solved. Two questions required the knowledge of the formula, knowledge of the procedure and performing of operations, e.g, the equation of a plane and find all square roots of a complex number (15% of the paper) (Group C). Use of an algorithm, e.g., finding the inverse of a matrix and converting a complex number in quotient form, solving simple linear equations with two unknowns made up 14% (Group D). Proofs made up 13% of the paper (Group E).

For Group A, the use of Cramer's rule is a good example to use. The student had to know the formulae to solve for x, y and z (D) which contains other concepts (the determinant and the determinants with respect to x, y and z. Needless to say if he/ she did not know the formulae he/ she could not answer the question. Not knowing how to calculate the determinants (P, C), the student again cannot solve the problem. Then another case with the dot-product when two vectors are given, and the student had to calculate the dot product. Student (14) did not know the correct formula (D) (e.g., he had divided by $||a||$ and not by $||a||^2$) but did know the procedure (P) and another student (5) who had to calculate the distance (D) between two points did not square the differences between the co-ordinates, but knew how to proceed to calculate the dot-product (P,C). Another student (2), although he/ she performed very well overall (85%) calculated the distance between two points as though the vectors started from the origin which could imply a careless mistake, or lack of conceptual knowledge.

It was noticed that in Sample A, most of the students that failed could hardly recall 20-30% of the formulae. Those that knew the formula (and how to calculate the other variables) achieved the highest marks which implies they possessed declarative, procedural, and conceptual knowledge. From sample B, the correct recall of formulae varied between 82% and 95% for this type of problems.

The results of this group support the ideas of a number of authors (Britton & Henderson, 2009; Donevska-Todorova, 2014; Skemp, 1987) that knowledge acquisition starts with the learning of concepts through initial abstractions which give rise to declarative knowledge. If these concepts are understood deep learning took place otherwise surface learning, which is accompanied by misconceptions (Luneta, 2015).

Understanding the relationships between the concepts and applying them implies that conceptual knowledge was acquired. Else misconceptions occur. Knowing how to perform operations between the concepts implies the student possesses procedural knowledge (Britton & Henderson, 2009; Donevska-Todorova, 2014; Krathwohl, 2002; Rittle-Johnson & Schneider, 2015; Soylu & Işik, 2008). The class performance in these sections was around 76% while the group performance (sample B) was around 84%.

For Group B, many students who could perform operations with 3x3 matrices/ determinants, when it came to 4x4 matrices/ determinants it appeared that what they thought if it was true for a 2x2 and 3x3 it will also be true for a 4x4. When the (square) determinants are especially of higher order (in this case a 4x4) in question 2b, where a 4x4 determinant was given many students failed to evaluate it. The student must also know alternative ways which could be shorter than expanding according to a row (column) (St). It was interesting to see only one student (2), to notice that the 4x4 determinant already contained 2 zeros in the first column and the subsequent 3x3 determinant also had two zeros. Most of the students opted for expanding rows. For example, student (8) performed row operations in the previous (Q1) correctly but with a 4x4 he/ she only operated with two columns the rest remained unchanged. Others applied the rules for a dot-product instead of the cross-product, while others obtained a vector from the dot product. In cases like these, it is evident that procedural knowledge in certain cases is not accompanied by conceptual knowledge thus one can say that procedural knowledge is in a fluid state. The class performance here was 72% while the group performance (sample B) was 80%. This above is another indication that may students lack either D or C or P

Groups C and D share the first part where an algorithm is similar to a procedure (Rittle-Johnson & Schneider, 2015). In the former case the student must know the algorithm how to convert a complex number in quotient form to standard form P and C) and in the latter how to construct the various vectors from given points to derive a formula for a plane in 3-D. Not knowing the procedure/ algorithm the student cannot answer the question. If the procedure is known, the student has to derive new information from the given information. The performance in these two sections was 90% for those that knew the algorithm or procedure and less than 10% for the rest. Of course, answers like the one given by student (4) where he/ she used the dot-product instead of the cross-product can confuse the researcher. It could be attributed to the fact that while the student had understood the dot-product he did not understand the cross-product (lack of C). For example, in question 3d where the question was to determine the equation of a plane given 3 points, and it requires the use of the cross-product he/ she also used the dot-product. How these two groups differ from the other two is that in Groups A and B, lack of declarative language leads to no solution to the problem. While in Groups C and D, lack of the procedure/ algorithm leads to no solution.

Finally, Group E deals with proofs (questions 2c, 2d). However even these proofs are not of high, abstract level as the questions were of the nature of a theorem in geometry. One does not need to understand the theorem, but you can still get full marks, if one can memorise the proof. Despite that, students as a whole performed worst (average 35%). In question 2c, the students had to prove if A is a matrix and if $A^2 = A$ prove that $\det(A) = 0$ or $\det(A)=1$as was stated above. The group average was 40%. For question 2d (group average 24%) students had to prove that two matrices are similar iff $A = PBP^{-1}$ where P is an invertible matrix and then prove that if A and B are similar then $\det(A) = \det(B)$. Looking at students' answers became apparent that most students confused the properties of matrices with those of determinants or used numerical examples to prove a general case. For example, student (6) for the first question (2c) makes the assumption that matrix A is the identity matrix. This assumption was made by 70% of students who attempted this question. Another student (2) uses A as a 2x2 matrix with all elements equal to 1 and $\det(A) = 1$. Student (8) states that since $A^2 = A$ then the matrices are identity matrices while student (13) uses an identity matrix and a matrix with elements equal to zeros and evaluates their determinant. Then in 2d, student (14) assumed that the all matrices are square matrices as a result it is possible that $PBP^{-1} = PP^{-1}B$ and the student cancels the Ps (as though $P.P^{-1} = P/P = 1$ and the student concludes A=B. Another student (3) (with exam mark of 77%) writes: Assume A=B then det (A)=det(B)!!! Student (2) (the second ranked with 81%) in the exam), multiplied both sides by P^{-1} and arrived at $P^{-1}A = BP^{-1}$ and concluded that A = B. Proofs here involve abstract conceptual knowledge mostly and some procedural knowledge, but all thought operations are abstractions on abstract objects. This falls in the algebraic mode of description (Donevska-Todorova, 2014; Hille, 2000).

From a linear algebra perspective, if we refer to the mode of description such as language and mode of thinking as identified in the literature (Çelik, 2015; Donevska-Todorova, 2014; Dorier & Sierpinska, 2002; Sierpinska, 2000) the content of the questions belong to the arithmetic mode and the required thinking is that of synthetic-arithmetic. However, vector spaces are involved here and are of very low level. The geometric mode features implicitly as students were not asked to represent any vector in geometric mode. Students who lacked conceptual knowledge showed that when they performed procedures such as row operations their approach was defective, though Çelik (2015) called them mistakes which in actual fact were misconceptions. However, as a whole where arithmetic thinking was involved the class performed well (76%) and the group (82%) compared to 68% of Çelik (2015).

The above discussion highlighted a number of problems that students encounter with acquiring the different types of knowledge which make the structure of knowledge of linear algebra. 73% of the paper followed the sequence of D→P→C (Groups A, B and C), 14% the sequence P→D→C (Group D) and C→D (Group E). This observation is in line with the literature (Krathwohl, 2002; Rittle-Johnson & Schneider, 2015; Shavelson et.al., 2005; Soylu & Işik, 2008; ;) where the *concepts-first* or *procedures-first* approach was used interchangeably (*iterative view*) on the types of knowledge where in some cases conceptual knowledge is developed from declarative and other cases from procedural. In abstract systems such as proofs, once understood, conceptual knowledge becomes declarative which is used to develop new conceptual knowledge through abstractions on abstract object (Rensaa & Vos, 2018; Skemo, 1987). The fact that success rate on the two proofs averaged the lowest (35%) is an indication that it is the most difficult part of linear algebra and was also highlighted by other researchers (Britton & Henderson, 2009; Dorier, 2002; Dorier & Sierpinska, 2002;).

Finally, using the types of knowledge to assess students' performance could be a valuable diagnostic tool that can be used for appropriate interventions. They can also be designed to improve pass rates. It gets its validity from the fact that this type of assessment correlates highly with the normal marking procedures. However, it should not be used as an assessment tool to assess examination papers because markers alone cannot be used to determine the different types of knowledge displayed by students.

CONCLUSION

The literature review has identified a number of factors that contribute to academic success in Linear Algebra. The studied literature indicated that conceptual and procedural knowledge is necessary to solve problems in Linear Algebra with little emphasis on declarative knowledge. Declarative knowledge forms the basis of the other two types. Schematic and strategic knowledge were not included as Linear Algebra problems require mostly declarative, procedural, and conceptual knowledge. This paper examined the types of knowledge as they are manifested in the final examination scripts of the students and the scripts were reassessed using the types of knowledge as an assessment instrument. Such an instrument can be of value to the teacher to improve his/ her teaching practice as it can be connected to Bloom's taxonomy as another dimension. Bloom's taxonomy is based on the depth of knowledge possessed by the student. By identifying the types of knowledge necessary to solve a problem (the length as suggested by Shavelson et al., 2005), it enables the teacher to be proactive and as a result students' misconceptions can be minimised. The analysis of students' answers indicated that students make too many assumptions (lack of conceptual knowledge). It can be concluded that although the examination paper contained predominantly declarative and conceptual knowledge, it cannot be divorced from the procedural knowledge in fact, declarative knowledge has been identified as the foundation of the other types, though it could be mere information acquired through surface learning. It was shown in the above analysis that lack of declarative knowledge leads to no solution to the problem. What distinguishes between the average and above the average students is the conceptual and procedural knowledge. And a final word for the lecturer: If you decide to teach procedures (procedural knowledge), ensure the procedures are understood (conceptual knowledge), while declarative knowledge must not be defective.

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